Step systems of Ptolemaic graphs and Some of its subclasses

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Abstract

Signpost systems are ternary relations T that encode path information in a strictly local manner. A triple $(u, x, v) \in T$ is interpreted as a "signpost" at vertex x to indicates that in order to reach v, a step from u to x is to be made [1]. In their most general form, they satisfy three natural axioms:

(A) If $(u, v, x) \in T$ then $(v, u, u) \in T$ for all $u, v, x \in V$.
(B) If $(u, v, x) \in T$ then $(v, u, x) \notin T$ for all $u, v, x \in V$
(H) If $u \neq v$ then there exists an $x \in V$ such that $(u, x, v) \in T$ for all $u, v \in V$.

The triples in $\boldsymbol{T}$ are called signposts. The underlying graph $\boldsymbol{G} = \boldsymbol{G_T}$ of $\boldsymbol{T}$ has vertex set $\boldsymbol{V}$ and edge set $\boldsymbol{E} = \boldsymbol{E(G)}$ such that, for all $\boldsymbol{u, v} \in \boldsymbol{V}$, holds

$$\boldsymbol{uv} \in \boldsymbol{E(G)} \iff (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{v}) \in \boldsymbol{T} \tag{1}$$

Step systems [2] are special signpost systems that were introduced in [2] to describe the geodesic structure of the underlying graph $\boldsymbol{G_T}$. As shown in [2], $\boldsymbol{T}$ is the set of all steps along shortest paths in $\boldsymbol{G_T}$ if and only if $\boldsymbol{T}$ is a signpost system satisfying

(C) If $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}) \in \boldsymbol{T}$ and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}) \in \boldsymbol{T}$ then $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \in \boldsymbol{T}$;
(D) If $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}) \in \boldsymbol{T}$ and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}) \in \boldsymbol{T}$ then $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{y}) \in \boldsymbol{T}$;
(F) If $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}) \in \boldsymbol{T}$, $(\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{y}) \in \boldsymbol{T}$, and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}) \in \boldsymbol{T}$ then $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \in \boldsymbol{T}$;
(G) If $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}) \in \boldsymbol{T}$ and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}) \in \boldsymbol{T}$ then $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \in \boldsymbol{T}$ or $(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{v}) \in \boldsymbol{T}$ or $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{y}) \in \boldsymbol{T}$;

In [2, 3] an additional "symmetry" axiom

(E) If $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{x}) \in \boldsymbol{T}$ and $(\boldsymbol{u}, \boldsymbol{y}, \boldsymbol{v}) \in \boldsymbol{T}$ then $\boldsymbol{y} = \boldsymbol{v}$

was invoked. This property, however, recently has been shown in [4] to be implied by the remaining seven axioms (A), (B), (C), (D), (F), (G) and (H).
Step systems of a variety of graph classes can be characterized by simple sets of axioms. Examples include trees [5], distance hereditary graphs [6], modular and median graphs, [1], bipartite graphs, partial cubes and weakly modular graphs [4].
In this paper, we characterize the step systems whose underlying graph is Ptolemaic as well as prominent sublcasses of Ptolemaic graphs, namely Laminar Chordal, Block Duplicate Graphs, Block Graphs, and AC graphs.

References

[1] Mulder HM, Nebeský L (2003) Modular and median signpost systems and their underlying graphs. Discuss Math Graph Theory 23:309–324, DOI 10.7151/dmgt.1204

[2] Nebeský L (1997) Geodesics and steps in a connected graph. Czechoslovak Math J 47(122):149–161, DOI 10.1023/A:1022404624515

[3] Nebeský L (2000) A theorem for an axiomatic approach to metric properties of graphs. Czechoslovak Math J 50:121–133, DOI 10.1023/A:1022401506441

[4] Changat M, JChavara J, Chitra MR, Mathew J, Stadler PF (2025) Step systems on graphs. Submitted

[5] Nebeský L (2006) Signpost systems and spanning trees of graphs. Czechoslovak Math J 56:885–893, DOI 10.1007/s10587-006-0063-z

[6] Nebeský L (2010) Distance hereditary graphs and signpost systems. Discrete Math 310:527–530, DOI 10.1016/j.disc.2009.03.032